Sliding-tile Puzzles

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The 15 Puzzle

Modern version
The tiles are held in the tray with a tongue-and-groove system.

Traditional version
The tiles are loose. They can be dumped out and replaced.
**Observation:**

From any starting arrangement, there is a sequence of moves that leads to either

\[ S \] or \[ S' \]

There is no sequence of moves that leads from \( S \) to \( S' \).
Conjecture:

An arrangement $T$ is reachable from the standard ordering only if $T$ represents an even permutation of $[1, 2, 3, 4, 5, \ldots, 15]$. 
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$[1, 2, 3, 4, 5, \ldots, 15]$.

Definitions:

An arrangement represents a permutation $[a_1, a_2, \ldots, a_{15}]$ when the numbers appear in that order, reading left to right, top to bottom, skipping over the blank.

$[4, 10, 15, 9, 6, 7, 12, 3, 11, 1, 8, 2, 5, 14, 13]$
Definitions:

A *move* consists of sliding an adjacent tile into the blank space.

An arrangement $T$ is *reachable* from an arrangement $S$ if there is some sequence of moves that transforms $S$ to $T$. Reachability is clearly an equivalence relation.

Observation:

Two arrangements that differ by a horizontal move represent the same permutation.

Only a vertical move changes the permutation.
Question:

What does a vertical move do to the permutation?
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**Answer:**

The permutation induced by a vertical move is always a 4-cycle.
Result:

An arrangement that is reachable from the standard starting arrangement represents

- an even permutation if the blank is in the second row or the fourth row; and
- an odd permutation if the blank is in the first row or the third row.

Corollary:

Any arrangement that has the blank in the bottom row and can be reached from the standard starting arrangement must represent an even permutation of $[1, 2, 3, \ldots, 15]$. 

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & \\
\end{array}
\]
**Harder question:** Can every even-permutation arrangement be reached from the standard starting position?
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**Answer:** Yes.

**Claim:**

Given any even permutation $\sigma$ of $[1, 2, \ldots, 15]$, there is a sequence of moves that transforms the standard starting position to an arrangement representing $\sigma$ with the blank in the lower-right corner.
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Proof sketch:
Lemma 1: (Textbook exercise) The alternating group $A_{15}$ is generated by

$$(1\ 2\ 3),\ (1\ 2\ 4),\ (1\ 2\ 5),\ \ldots,\ (1\ 2\ 15).$$

Lemma 1': The alternating group $A_{15}$ is generated by

$$(13\ 14\ 15),\ (12\ 14\ 15),\ (11\ 14\ 15),\ \ldots,\ (1\ 14\ 15).$$
Proof sketch, continued:

**Lemma 2:** For an arrangement with the blank square in the bottom row, there is a sequence of moves that cyclically permutes the three tiles in the bottom row, and leaves all other tiles fixed.
Proof sketch, continued:

**Lemma 3**: For an arrangement with the blank square in the bottom row, there is a sequence of moves that leaves the two rightmost tiles in the bottom row fixed and replaces the remaining bottom-row tile with any other tile.

Idea: Walk the blank around a non-self-intersecting closed path enough times to advance the target tile into the “13” position.
Proof sketch, conclusion:

1. By Lemma 3, we can move any tile $x$ other than 14 or 15 into the 13 spot.

2. Using Lemma 2, we cyclically permute the three tiles in the bottom row.

3. We reverse the moves we made in (1). This returns all tiles except 14, 15, and $x$ to their original positions. We have effected the permutation $(x\ 14\ 15)$.

4. By Lemma 1', the set of permutations $\{(x\ 14\ 15) : x = 1, 2, \ldots, 13\}$ generates the alternating group $A_{15}$. 

History of the 15 Puzzle:

The 15 Puzzle, first appearing in late 1879, became a craze throughout the first quarter of 1880.

In February and March 1880, articles and letters about the 15 Puzzle appeared almost daily in newspapers from Boston to Chicago and beyond.

The novelty:

With the tiles placed haphazardly, the puzzle is sometimes (relatively) easy, and sometimes impossible.
The Diabolical Invention of Some Enemy of Mankind.

... A gentleman saw one in the store, and it looked so simple that he took it home to amuse the children. In ten minutes ... he was oblivious to all outward things, and went on, hour after hour, moving the little blocks of wood with the feverish intensity of a madman. ...

... To-day there is hardly a pleasant home in the city that has not the dark shadow of “15” across its threshold....

... Occasionally some one will get the fifteen numbers in the proper order, but his elation is short lived. To save him he cannot tell how he did it, nor can he do it again. All theories are wrong and experience is of no avail....
More history:

About a decade after the 15-puzzle craze, the famous puzzlist Sam Loyd let it be known that he had invented the puzzle. He is credited with its invention in

*The Dictionary of American Biography*
*The Encyclopedia Britannica*
*Scientific American* columns by Martin Gardner
*Mathematics papers by Archer and Spitznagel...
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In 2006, Slocum and Sonneveld showed that Loyd did *not* invent the 15 puzzle (or Parchesi or “Pigs in Clover”), and that the actual inventor was probably Noyes P. Chapman, who was the postmaster in Canastota, New York.
Hans Liebeck’s quarter-turn challenge

Answer
Clockwise: Yes
Counterclockwise: No
Variations:
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This suggests a card trick . . .
Card Trick:

Lay out eleven cards in a 3-by-4 array, in order, on a tray.
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Lay out eleven cards in a 3-by-4 array, in order, on a tray.

Make some sequence of moves to scramble the cards.

Hand the tray to your victim (Liebeck’s word), giving it a half-turn as you do so.

Watch your victim try to restore the cards to their natural order.
References:

Solutions


History


Variations, Card Tricks, and so on